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Five-dimensional supersymmetric Chern-Simons action as a hypermultiplet quantum correction

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Abstract

Building on the covariant supergraph techniques in 4D $\mathcal{N}=2$ harmonic superspace, we develop a manifestly 5D $\mathcal{N}=1$ supersymmetric and gauge covariant formalism to compute the one-loop effective action for a hypermultiplet coupled to a background vector multiplet. As a simple application, we demonstrate the generation of a supersymmetric Chern-Simons action at the quantum level, both in the Coulomb and the non-Abelian phases. These superfield results are in agreement with the earlier component considerations of Seiberg et al. Our analysis suggests that similar calculations in terms of hybrid 4D superfields or within the 5D projective superspace approach may allow one to extract suitable formulations for the non-Abelian 5D supersymmetric Chern-Simons theory.

In memory of Steve Irwin

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Five-dimensional supersymmetric theories with eight supercharges have recently attracted much attention, mainly in the context of brane-world scenarios. Phenomenological applications favour a hybrid superspace formulation for these theories in which one keeps manifest only the 4D $\mathcal{N}=1$ supersymmetry, in the spirit of Marcus, Sagnotti and Siegel [1]. A number of 5D rigid supersymmetric models have been constructed in such a setting [2, 3, 4]. Many of them have also been recast in 5D superspace [5, 6] (where some new models have also been put forward).

It seems robust to view the hybrid and the 5D manifestly supersymmetric settings as complimentary to each other. In the hybrid approach, it is an open interesting problem to construct a non-Abelian 5D supersymmetric Chern-Simons action.[1] In the 5D harmonic superspace approach, such an action was constructed several years ago in (the erratum of) [8]. Unfortunately, it is not trivial to reduce the harmonic superspace construction of [8] to the hybrid formalism. Still, it is natural to wonder whether we can make use of the construction in [8] to get any practical information, even indirect, about the explicit structure of the non-Abelian 5D supersymmetric Chern-Simon action formulated in terms of 4D $\mathcal{N}=1$ superfields. One of the aims of this note is to give a positive answer. Ten years ago, it was demonstrated at the component level [9, 10, 11] that, in five dimensions, a Chern-Simons term is generated by integrating out massive hypermultiplets. In this note, we follow the ideas put forward in [9, 10, 11] and carry out explicit one-loop harmonic superspace calculations, both along the Coulomb branch and in the non-Abelian phase, and demonstrate that Zupnik's action [8] appears as a leading quantum correction. Therefore, if one would repeat the same calculations within the hybrid superspace formulation, or within the projective superspace formulation, it should be possible to extract a non-Abelian 5D supersymmetric Chern-Simons action from the low-energy effective action.

From a more general perspective, this note is aimed at developing covariant background field techniques for computing quantum corrections in 5D $\mathcal{N}=1$ supersymmetric gauge theories. Although there have appeared several hybrid superspace calculations of various quantum effects, see [12, 13] and references therein, covariant 5D supergraph techniques seem to be completely unexplored, although the properties of 5D $\mathcal{N}=1$ supersymmetric gauge theories are quite interesting [9, 10, 11]. Unlike the five-dimensional case, powerful covariant supergraph methods have been developed for the 4D $\mathcal{N}=2$ super Yang-Mills theories [14] (see [15] for a review) and [16], and here we will build on those results.

[1] Such an action has been given only in the Wess-Zumino gauge [7].

The classical action for a massless hypermultiplet coupled to a background 5D $\mathcal{N}=1$ vector multiplet is

$$S_{\rm hyper} \;=\; -\int {\rm d}\zeta^{(-4)}\, \breve{q}^{+}\mathcal{D}^{++}q^{+}\ , \tag{1}$$

with $\mathcal{D}^{++}=D^{++}+{\rm i}\,\mathcal{V}^{++}$ the analyticity-preserving covariant derivative, and $\mathcal{V}^{++}(\zeta)$ is the analytic prepotential containing all the information about the off-shell vector multiplet [17]. The dynamical variable $q^{+}(\zeta)$ is a covariantly analytic superfield of harmonic U(1) charge $+1$, $\mathcal{D}^{+}_{\hat\alpha}q^{+}=0$, and $\breve{q}^{+}$ is the conjugate of q^{+} with respect to the analyticity-preserving conjugation [17]. The integration in (1) is carried out over the analytic subspace of the harmonic superspace $\mathbb{R}^{5|8}\times S^{2}$, see [5] for more details and our 5D notation and conventions.

The hypermultiplet effective action reads

$$\Gamma_{\rm hyper}={\rm i}\,{\rm Tr}\,\ln\mathcal{D}^{++}=-{\rm i}\,{\rm Tr}\,\ln G^{(1,1)}\ , \tag{2}$$

with $G^{(1,1)}(\zeta_1,\zeta_2)$ the hypermultiplet Green function (compare with the four-dimensional case [14]):

$$\begin{aligned}\mathcal{D}_1^{++}G^{(1,1)}(\zeta_1,\zeta_2) &= \delta_A^{(3,1)}(\zeta_1,\zeta_2)\ ,\\ G^{(1,1)}(\zeta_1,\zeta_2) &= -\frac{1}{\widehat{\Box}_1}(\hat{\mathcal{D}}_1^{+})^4\,(\hat{\mathcal{D}}_2^{+})^4\mathbb{1}\delta^{13}(z_1-z_2)\frac{1}{(u_1^{+}u_2^{+})^3}\ .\end{aligned} \tag{3}$$

Here $\delta^{13}(z-z')=\delta^{5}(x-x')\,\delta^{8}(\theta-\theta')$ is the delta-function in the conventional superspace, $\delta_A^{(3,1)}(\zeta_1,\zeta_2)$ the appropriate covariantly analytic delta-function,

$$\begin{aligned}\delta_A^{(3,1)}(\zeta,\zeta') &= (\hat{\mathcal{D}}^{+})^4\,\mathbb{1}\,\delta^{13}(z-z')\,\delta^{(-1,1)}(u,u')\ ,\\ (\hat{\mathcal{D}}^{+})^4 &= -\frac{1}{32}(\hat{\mathcal{D}}^{+})^2\,(\hat{\mathcal{D}}^{+})^2\ , \qquad (\hat{\mathcal{D}}^{+})^2=\mathcal{D}^{+\hat\alpha}\mathcal{D}^{+}_{\hat\alpha}\ ,\end{aligned} \tag{4}$$

and $(u_1^{+}u_2^{+})^{-3}$ a special harmonic distribution [17].

In eq. (3), $\widehat{\Box}$ is the covariantly analytic d'Alembertian [5]

$$\widehat{\Box}=\mathcal{D}^{\hat a}\mathcal{D}_{\hat a}+(\mathcal{D}^{+\hat\alpha}\mathcal{W})\,\mathcal{D}^{-}_{\hat\alpha}-\frac{1}{4}(\hat{\mathcal{D}}^{+\hat\alpha}\mathcal{D}^{+}_{\hat\alpha}\mathcal{W})\,\mathcal{D}^{--}+\frac{1}{4}(\mathcal{D}^{+\hat\alpha}\mathcal{D}^{-}_{\hat\alpha}\mathcal{W})-\mathcal{W}^2\ . \tag{5}$$

Given a covariantly analytic superfield φ, $\mathcal{D}^{+}_{\hat\alpha}\varphi=0$, the identity

$$\widehat{\Box}\,\varphi=\frac{1}{2}(\hat{\mathcal{D}}^{+})^4(\mathcal{D}^{--})^2\varphi$$

holds, and therefore $\widehat{\Box}$ preserves analyticity, $\mathcal{D}^{+}_{\hat\alpha}\,\widehat{\Box}\,\varphi=0$. To prove the above identity, one should make use of the following properties of the 5D $\mathcal{N}=1$ gauge-covariant derivatives in harmonic superspace[2] [8, 5]

$$
\begin{aligned}
\{\mathcal{D}^+_{\hat\alpha}\,, \mathcal{D}^-_{\hat\beta}\} &= 2\,\mathrm{i}\,\mathcal{D}_{\hat\alpha\hat\beta} - 2\,\varepsilon_{\hat\alpha\hat\beta}\,\mathcal{W}\ , \\
\big[\mathcal{D}^+_{\hat\gamma}\,, \mathcal{D}_{\hat\alpha\hat\beta}\big] &= \mathrm{i}\,\Big(\varepsilon_{\hat\alpha\hat\beta}\,\mathcal{D}^+_{\hat\gamma} + 2\varepsilon_{\hat\gamma\hat\alpha}\,\mathcal{D}^+_{\hat\beta} - 2\varepsilon_{\hat\gamma\hat\beta}\,\mathcal{D}^+_{\hat\alpha}\Big)\mathcal{W}\ , \\
\big[\mathcal{D}^{++}\,, \mathcal{D}^-_{\hat\alpha}\big] &= \mathcal{D}^+_{\hat\alpha}\ , \qquad\qquad \big[\mathcal{D}^{++}\,, \mathcal{D}^+_{\hat\alpha}\big] = 0\ , \\
\big[\mathcal{D}^{--}\,, \mathcal{D}^+_{\hat\alpha}\big] &= \mathcal{D}^-_{\hat\alpha}\ , \qquad\qquad \big[\mathcal{D}^{--}\,, \mathcal{D}^-_{\hat\alpha}\big] = 0\ .
\end{aligned} \tag{6}
$$

The field strength $\mathcal{W}$ obeys the Bianchi identity

$$
\mathcal{D}^+_{\hat\alpha}\mathcal{D}^+_{\hat\beta}\mathcal{W} = \frac{1}{4}\varepsilon_{\hat\alpha\hat\beta}\,(\hat{\mathcal{D}}^+)^2\mathcal{W} \quad\Rightarrow\quad \mathcal{D}^+_{\hat\alpha}\mathcal{D}^+_{\hat\beta}\mathcal{D}^+_{\hat\gamma}\mathcal{W} = 0\ . \tag{7}
$$

It should be mentioned that the action (1) is given in the so-called λ-representation [17] in which the gauge-covariant derivatives $\mathcal{D}^+_{\hat\alpha}$ possess no connection, i.e. $\mathcal{D}^+_{\hat\alpha}$ coincide with the rigid spinor derivatives $D^+_{\hat\alpha}$. The explicit expression for the Green function $G^{(1,1)}(\zeta_1,\zeta_2)$, eq. (3), is given in the τ-frame [17] (in the λ-frame, the Green function involves the bridge superfield at two superspace points [14]). The τ-frame is characterised by the properties that (i) the harmonic gauge-covariant derivatives $\mathcal{D}^{++}$ and $\mathcal{D}^{--}$ possess no connection, i.e. $\mathcal{D}^{\pm\pm} = D^{\pm\pm}$; and (ii) the spinor derivatives $\mathcal{D}^+_{\hat\alpha}$ and $\mathcal{D}^-_{\hat\alpha}$ are expressed in terms of the *harmonic-independent* gauge-covariant derivatives [18],

$$
\mathcal{D}_{\hat A} = (\mathcal{D}_{\hat a}, \mathcal{D}^i_{\hat\alpha}) = D_{\hat A} + \mathrm{i}\,\mathcal{V}_{\hat A}(z)\ , \qquad [\mathcal{D}_{\hat A}\,, \mathcal{D}_{\hat B}\} = T_{\hat A\hat B}{}^{\hat C}\,\mathcal{D}_{\hat C} + \mathrm{i}\,\mathcal{F}_{\hat A\hat B}\ , \tag{8}
$$

as follows: $\mathcal{D}^+_{\hat\alpha} = \mathcal{D}^i_{\hat\alpha}\,u^+_i$ and $\mathcal{D}^-_{\hat\alpha} = \mathcal{D}^i_{\hat\alpha}\,u^-_i$. Here $D_{\hat A} = (\partial_{\hat a}, D^i_{\hat\alpha})$ are the flat covariant derivatives obeying the anti-commutation relations $[D_{\hat A}, D_{\hat B}\} = T_{\hat A\hat B}{}^{\hat C}\,D_{\hat C}$.

The above definition of $\Gamma_{\rm hyper}$ can be seen to be purely formal, since the operator $\mathcal{D}^{++}$ maps analytic superfields q^+ with U(1) charge +1 to analytic superfields of U(1) charge +3, see also [16]. However, the expression for an arbitrary variation of the effective action

$$
\delta\Gamma_{\rm hyper} = -\mathrm{Tr}\,\Big\{\delta\mathcal{V}^{++}\,G^{(1,1)}\Big\} \tag{9}
$$

can be made well-defined. Using Schwinger's proper-time representation [19], we introduce a regularized variation of the effective action

$$
\begin{aligned}
\delta\Gamma_{{\rm hyper},\epsilon} &= \mathrm{tr}\int \mathrm{d}\zeta^{(-4)}\,\delta\mathcal{V}^{++}\langle J^{++}_\epsilon\rangle_\lambda\ , \qquad\qquad \mathcal{D}^+_{\hat\alpha}\langle J^{++}_\epsilon\rangle = 0\ , \\
\langle J^{++}_\epsilon\rangle_\tau &= \int_0^\infty \mathrm{d}(\mathrm{i}s)\,(\mathrm{i}\,\mu^2 s)^\epsilon\,\mathrm{e}^{\mathrm{i}\,s\,\widehat{\Box}_1}\,(\hat{\mathcal{D}}^+_1)^4\,(\hat{\mathcal{D}}^+_2)^4\,\frac{\mathbb{1}\,\delta^{13}(z_1-z_2)}{(u^+_1 u^+_2)^3}\Big|_{1=2}\ ,
\end{aligned} \tag{10}
$$

[2]These properties follow from the 5D $\mathcal{N}=1$ vector multiplet formulation [18] in conventional superspace $\mathbb{R}^{5|8}$ parametrized by coordinates $z^{\hat A} = (x^{\hat a}, \theta^{\hat\alpha}_i)$.

with ϵ the regularization parameter, $\epsilon \to 0$ upon renormalization, and μ the normalization point. Gauge invariance of the effective action is equivalent to the fact that $\langle J^{++}_\epsilon \rangle$ is a conserved current,

$$\mathcal{D}^{++} \langle J^{++}_\epsilon \rangle = 0 \ . \tag{11}$$

The second line in (10) can be brought to a more useful form by applying the identity

$$\begin{aligned}(\hat{\mathcal{D}}^+_1)^4 (\hat{\mathcal{D}}^+_2)^4 \, \frac{\mathbb{1} \delta^{13}(z_1 - z_2)}{(u^+_1 u^+_2)^q} &= (\hat{\mathcal{D}}^+_1)^4 \left\{ (\hat{\mathcal{D}}^-_1)^4 \, \frac{1}{(u^+_1 u^+_2)^{q-4}} - \frac{1}{4} \, \Delta^{--}_1 \, \frac{(u^-_1 u^+_2)}{(u^+_1 u^+_2)^{q-3}} \right. \\ &\left. - \widehat{\Box}_1 \, \frac{(u^-_1 u^+_2)^2}{(u^+_1 u^+_2)^{q-2}} - \frac{1}{4} (q-3) \, (\mathcal{D}^+_1 \mathcal{D}^+_1 \mathcal{W}_1) \, \frac{(u^-_1 u^+_2)^3}{(u^+_1 u^+_2)^{q-1}} \right\} \mathbb{1} \, \delta^{13}(z_1 - z_2) \ ,\end{aligned} \tag{12}$$

with q an integer, and therefore

$$\begin{aligned}(\hat{\mathcal{D}}^+_1)^4 (\hat{\mathcal{D}}^+_2)^4 \, \frac{\mathbb{1} \delta^{13}(z_1 - z_2)}{(u^+_1 u^+_2)^3} &= (\hat{\mathcal{D}}^+_1)^4 \left\{ (\hat{\mathcal{D}}^-_1)^4 \, (u^+_1 u^+_2) - \frac{1}{4} \, \Delta^{--}_1 \, (u^-_1 u^+_2) \right. \\ &\left. - \widehat{\Box}_1 \, \frac{(u^-_1 u^+_2)^2}{(u^+_1 u^+_2)} \right\} \mathbb{1} \, \delta^{13}(z_1 - z_2) \ .\end{aligned} \tag{13}$$

Here

$$\Delta^{--} = \mathrm{i} \, \mathcal{D}^{\hat{\alpha}\hat{\beta}} \mathcal{D}^-_{\hat{\alpha}} \mathcal{D}^-_{\hat{\beta}} + \mathcal{W} (\hat{\mathcal{D}}^-)^2 + 4 (\mathcal{D}^{-\hat{\alpha}} \mathcal{W}) \mathcal{D}^-_{\hat{\alpha}} + (\mathcal{D}^- \mathcal{D}^- \mathcal{W}) \ . \tag{14}$$

The identity (12) is a five-dimensional analogue of the one obtained in [16]. It can be derived by using the anti-commutation relations (6).

Similar to the four-dimensional case [14], the operator $\widehat{\Box}$ possesses the property

$$(\hat{\mathcal{D}}^+)^4 \, \widehat{\Box} = \widehat{\Box} \, (\hat{\mathcal{D}}^+)^4 \ . \tag{15}$$

Also in complete analogy with the four-dimensional case [16], one can show that the third term in (13) does not contribute to $\langle J^{++}_\epsilon \rangle$ in the limit $\epsilon \to 0$. Therefore, the current in (10) can be rewritten as follows[3]

$$\begin{aligned}\langle J^{++}_\epsilon \rangle_\tau = \int_0^\infty & \mathrm{d}(\mathrm{i}s) \, (\mathrm{i} \, \mu^2 s)^\epsilon \, (\hat{\mathcal{D}}^+)^4 \, \mathrm{e}^{\mathrm{i} \, s \, \widehat{\Box}} \\ & \times \left\{ (u^+ u'^+) \, (\hat{\mathcal{D}}^-)^4 + \frac{1}{4} \, \Delta^{--} \right\} \mathbb{1} \, \delta^{13}(z - z') \Big|_{z=z', \, u=u'} \ ,\end{aligned} \tag{16}$$

where we have use the indentity $(u^+ u^-) = -(u^- u^+) = 1$.

[3] By construction, the first line of eq. (10) is given in the λ-representation. When computing $\langle J^{++}_\epsilon \rangle$ using eq. (16), we switch from the λ-frame to the τ-frame.

At this stage, it is useful to introduce, following the general approach developed in [20], a new representation for the full delta-function

$$\mathbb{1}\,\delta^{13}(z-z') = I(z,z')\,\delta^5(x-x')\,\delta^8(\theta-\theta') = I(z,z')\int \frac{\mathrm{d}^5k}{(2\pi)^5}\,\mathrm{e}^{\mathrm{i}\,k_{\hat a}\rho^{\hat a}}\,\delta^8(\Theta)\ , \tag{17}$$

where

$$\xi^{\hat A} \equiv \xi^{\hat A}(z,z') = -\xi^{\hat A}(z',z) = \begin{cases} \rho^{\hat a} = (x-x')^{\hat a} + \mathrm{i}\,\theta^{\hat\alpha}_i(\Gamma^{\hat a})_{\hat\alpha\hat\beta}\,\theta'^{\hat\beta i}\ , \\ \Theta^{\hat\alpha}_i = (\theta-\theta')^{\hat\alpha}_i \end{cases} \tag{18}$$

is the supersymmertic two-point function, and $I(z,z')$ stands for the parallel displacement propagator along the straight line connecting the points z and z'. The latter is a unique two-point function, which takes its values in the gauge group and which obeys the first-order differential equation equation and special boundary condition

$$\xi^{\hat A}\mathcal{D}_{\hat A}\,I(z,z') = \xi^{\hat A}\Big(D_{\hat A} + \mathrm{i}\,\mathcal{V}_{\hat A}(z)\Big)I(z,z') = 0\ , \qquad I(z,z) = \mathbb{1}\ . \tag{19}$$

These imply the important relation

$$I(z,z')\,I(z',z) = \mathbb{1}\ , \tag{20}$$

and also the equation at z'

$$\xi^{\hat A}\mathcal{D}'_{\hat A}\,I(z,z') = \xi^{\hat A}\Big(D'_{\hat A}\,I(z,z') - \mathrm{i}\,I(z,z')\,\mathcal{V}_{\hat A}(z')\Big) = 0\ . \tag{21}$$

One of the fundamental properties of $I(z,z')$ [20] is

$$\begin{aligned}\mathcal{D}_{\hat B}I(z,z') = \mathrm{i}\,I(z,z')\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\Big\{&n\,\xi^{\hat A_n}\ldots\xi^{\hat A_1}\mathcal{D}'_{\hat A_1}\ldots\mathcal{D}'_{\hat A_{n-1}}\mathcal{F}_{\hat A_n\,\hat B}(z') \\ &+\frac{1}{2}(n-1)\,\xi^{\hat A_n}T_{\hat A_n\,\hat B}{}^{\hat C}\,\xi^{\hat A_{n-1}}\ldots\xi^{\hat A_1}\mathcal{D}'_{\hat A_1}\ldots\mathcal{D}'_{\hat A_{n-2}}\mathcal{F}_{\hat A_{n-1}\,\hat C}(z')\Big\}\ .\end{aligned} \tag{22}$$

Together with the identity

$$\mathcal{D}^i_{\hat\alpha}\,\rho^{\hat a} = -\mathrm{i}\,(\Gamma^{\hat a})_{\hat\alpha\hat\beta}\,\Theta^{\hat\beta i} \quad\longrightarrow\quad \xi^{\hat B}\,\mathcal{D}_{\hat B}\xi^{\hat A} = \xi^{\hat A}\ , \tag{23}$$

eq. (22) allows us to compute the integrand in (16) in a manifestly covariant way, as a series in powers of the field strength and its covariant derivatives.[4] As a first step, one pushes the plane wave $\exp(\mathrm{i}\,k_{\hat a}\rho^{\hat a})$ through all the operatorial factors in (16) to the left,

[4] Somewhat different techniques, valid specifically for one-loop calculations, were suggested in [21].

and then it turns into unity in the coincidence limit. This has the following impact on the covariant derivatives

$$\mathcal{D}_{\hat{a}} \;\to\; \mathcal{D}_{\hat{a}} + \mathrm{i}\, k_{\hat{a}} \;, \qquad \mathcal{D}^i_{\hat{\alpha}} \;\to\; \mathcal{D}^i_{\hat{\alpha}} + k_{\hat{a}}\, (\Gamma^{\hat{a}})_{\hat{\alpha}\hat{\beta}}\, \Theta^{\hat{\beta} i} \;. \tag{24}$$

Then, the momentum integration reduces to doing Gaussian integrals

$$\frac{1}{(2\pi)^5} \int \mathrm{d}^5 k\, \mathrm{e}^{-\mathrm{i}\, s\, k^2}\, s k^{\hat{a}_1} \ldots s k^{\hat{a}_n} = \frac{1}{(4\pi s)^{5/2}} \int \mathrm{d}^5 k\, \mathrm{e}^{-\mathrm{i}\, k^2}\, \sqrt{s} k^{\hat{a}_1} \ldots \sqrt{s} k^{\hat{a}_n} \;, \tag{25}$$

and this is a textbook problem. Finally, the covariant derivatives in (16) should hit either $\xi^{\hat{A}}$ (this is easy) or the parallel displacement propagator, and then eq. (22) applies.

We are now prepared to compute the effective action. Consider first the Coulomb branch of the theory, where the gauge field takes its values in the Cartan subalgebra. Let us denote $\widehat{\Box} = \mathcal{O} - \mathcal{W}^2$. Then

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\, s\, \widehat{\Box}} &= \mathrm{e}^{-\mathrm{i}\, s\, \mathcal{W}^2}\, \mathrm{e}^{\mathrm{i}\, s\, \mathcal{O}} \;+\; O(\mathcal{D}\, \mathcal{W}^2) \qquad\qquad (26)\\ &= \mathrm{e}^{-\mathrm{i}\, s\, \mathcal{W}^2} \mathrm{e}^{\mathrm{i}\, s \mathcal{D}^{\hat{a}} \mathcal{D}_{\hat{a}}} \Big\{ 1 - \frac{\mathrm{i}\, s}{4} (\mathcal{D}^+ \mathcal{D}^+ \mathcal{W})\, \mathcal{D}^{--} + \frac{(\mathrm{i}\, s)^2}{2}\, (\mathcal{D}^{+\hat{\beta}} \mathcal{W})(\mathcal{D}^{+\hat{\alpha}} \mathcal{W})\, \mathcal{D}^-_{\hat{\alpha}} \mathcal{D}^-_{\hat{\beta}} \Big\} + \ldots \end{aligned}$$

The second term in the curly brackets produces a non-vanishing result when hitting $(u^+ u'^+)$ in (16),

$$\mathcal{D}^{--} (u^+ u'^+)|_{u=u'} = (u^- u'^+)|_{u=u'} = -1 \;.$$

The third term in the curly brackets can produce a non-vanishing contribution to (16) only if paired with $\mathcal{W}(\hat{\mathcal{D}}^-)^2$ in Δ^{--}, due to the dentity $\mathcal{D}^-_{\hat{\alpha}} \mathcal{D}^-_{\hat{\beta}} (\hat{\mathcal{D}}^-)^2 = -8\, \varepsilon_{\hat{\alpha}\hat{\beta}}\, (\hat{\mathcal{D}}^-)^4$.

The result of calculation is

$$\langle J^{++}_\epsilon \rangle = \mathrm{i} \int_0^\infty \mathrm{d}s\, \frac{(\mu^2 s)^\epsilon}{(4\pi s)^{5/2}} \Big\{ \frac{1}{4} (\mathcal{D}^+ \mathcal{D}^+ \mathcal{W})\, s - \mathcal{W}\, (\mathcal{D}^{+\hat{\alpha}} \mathcal{W})(\mathcal{D}^+_{\hat{\alpha}} \mathcal{W})\, s^2 \Big\}\, \mathrm{e}^{-s \mathcal{W}^2} + \ldots \tag{27}$$

Using the identities $\Gamma(1/2) = (-1/2)\, \Gamma(-1/2) = \sqrt{\pi}$, we obtain in the limit $\epsilon \to 0$

$$\langle J^{++} \rangle = -\mathrm{sign}(\mathcal{W})\, \frac{1}{2(4\pi)^2}\, \mathcal{G}^{++} + \ldots \;, \tag{28}$$

where $\mathcal{G}^{++}$ denotes the covariantly analytic descendant of $\mathcal{W}$ introduced in [5]

$$-\mathrm{i}\, \mathcal{G}^{++} = \mathcal{D}^{+\hat{\alpha}} \mathcal{W}\, \mathcal{D}^+_{\hat{\alpha}} \mathcal{W} + \frac{1}{4} \{ \mathcal{W} \,, (\hat{\mathcal{D}}^+)^2 \mathcal{W} \} \;, \qquad \mathcal{D}^+_{\hat{\alpha}} \mathcal{G}^{++} = \mathcal{D}^{++} \mathcal{G}^{++} = 0 \;. \tag{29}$$

This expression for $\mathcal{G}^{++}$ holds for the general non-Abelian vector multilpet. When the vector multiplet is restricted to the Cartan subalgebra, i.e. the case we are currently discussing, the anticommutator $\{\mathcal{W} \,, (\hat{\mathcal{D}}^+)^2 \mathcal{W}\}$ in (29) reduces to $2\, \mathcal{W} (\hat{\mathcal{D}}^+)^2 \mathcal{W}$.

In the U(1) case, the supersymmetric Chern-Simons action, which was first constructed in [8] in terms of the prepotential $\mathcal{V}^{++}$, can be represented in the form [5]

$$S_{\rm CS} = \frac{1}{3} \int {\rm d}\zeta^{(-4)}\, \mathcal{V}^{++}\, \mathcal{G}^{++}\ . \tag{30}$$

Varying $S_{\rm CS}$ gives

$$\delta S_{\rm CS} = \int {\rm d}\zeta^{(-4)}\, \delta\mathcal{V}^{++}\, \mathcal{G}^{++}\ . \tag{31}$$

Comparing with (28) we see that the leading quantum correction computed indeed coincides with a sum of super Chern-Simons actions associated with all the U(1) factors in the Coulomb branch.

In the non-Abelian case, the supersymmetric Chern-Simons action will be defined to vary as follows:

$$\delta S_{\rm CS} = {\rm tr} \int {\rm d}\zeta^{(-4)}\, \delta\mathcal{V}^{++}\, \mathcal{G}^{++}\ , \tag{32}$$

similarly to (31). This definition is equivalent to the one originally given in [8]. Indeed, in the λ-frame the field strength $\mathcal{W}$ has a simple expression in terms of $\mathcal{V}^{--}$

$$\mathcal{W}_\lambda = \frac{\rm i}{8} (\hat{D}^+)^2 \mathcal{V}^{--}\ , \tag{33}$$

and this can be used to show that

$$\mathcal{G}^{++} = \frac{1}{2} (\hat{D}^+)^4 \{\mathcal{W}_\lambda, \mathcal{V}^{--}\}\ . \tag{34}$$

Then, eq. (32) can be rewritten

$$\delta S_{\rm CS} = \frac{1}{2}\, {\rm tr} \int {\rm d}^{13}z\, {\rm d}u\, \delta\mathcal{V}^{++}\, \{\mathcal{W}_\lambda, \mathcal{V}^{--}\}\ , \tag{35}$$

what coincides with the definition given in [8]. Zupnik has integrated the variation (35) and derived $S_{\rm CS}$ as an infinite series in powers of the prepotential $\mathcal{V}^{++}$. This series terminates if one chooses a standard Wess-Zumino gauge for the vector multiplet, and then the action can be readily reduced to components.

Instead of giving the explicit expression for $S_{\rm CS}$ in terms of the prepotential, let us simply demonstrate the integrability of (35). Consider a second variation

$$\delta_2 \delta_1 S_{\rm CS} = \ {\rm tr} \int {\rm d}^{13}z\, {\rm d}u\, \delta_1\mathcal{V}^{++}\, \{\delta_2\mathcal{V}^{--}, \mathcal{W}_\lambda\}\ , \tag{36}$$

and transform it into the τ-frame

$$\begin{aligned}\delta_2 \delta_1 S_{\rm CS} &= {\rm tr} \int {\rm d}^{13}z\, {\rm d}u\, (\delta_1\mathcal{V}^{++})_\tau\, \{(\delta_2\mathcal{V}^{--})_\tau, \mathcal{W}\} \\ &= {\rm tr} \int {\rm d}^{13}z\, {\rm d}u\, \mathcal{W}\, \{(\delta_1\mathcal{V}^{++})_\tau\, , (\delta_2\mathcal{V}^{--})_\tau\}\ . \end{aligned} \tag{37}$$

In the λ-frame, the variations δV^{++} and δV^{--} are related to each other as follows [22]

$$\mathcal{D}^{++}\delta\mathcal{V}^{--} = \mathcal{D}^{--}\delta\mathcal{V}^{++} \ . \tag{38}$$

In the τ-frame, this becomes

$$D^{++}(\delta\mathcal{V}^{--})_\tau = D^{--}(\delta\mathcal{V}^{++})_\tau \ . \tag{39}$$

This equation is known to have the following solution [17, 22]

$$(\delta\mathcal{V}^{--})_\tau(u) = \int \mathrm{d}u_1 \, \frac{(\delta\mathcal{V}^{++})_\tau(u_1)}{(u^+u_1^+)^2} \ . \tag{40}$$

Using this result in (37) and taking into account the fact that $\mathcal{W}$ is harmonic-independent, in the τ-frame, we conclude

$$\delta_2\delta_1 S_{\rm CS} = \delta_1\delta_2 S_{\rm CS} \ . \tag{41}$$

Now, let us turn to the consideration of a massive hypermultiplet transforming in an arbitrary representation of the gauge group G. One can use the same action (1) provided one assumes that (i) the gauge group is $G \times \mathrm{U}(1)$, and (ii) the U(1) gauge field $\mathcal{V}_0^{++}$ possesses a constant field strength $\mathcal{W}_0 = \text{const}$, $|\mathcal{W}_0| = m$, see [23] for more details. This effectively amounts to replacing

$$\mathcal{V}^{++} \ \to \ \tilde{\mathcal{V}}^{++} = \mathcal{V}_0^{++} + \mathcal{V}^{++} \ , \qquad \mathcal{W} \ \to \ \tilde{\mathcal{W}} = \mathcal{W}_0 + \mathcal{W} \tag{42}$$

in most of the above expressions. Of course, we should also modify the gauge covariant derivatives and the field strength $\mathcal{F}_{\hat{A}\hat{B}}$ in (8) similarly. The U(1) gauge field $\mathcal{V}_0^{++}$ is completely frozen, and therefore eq. (9) involves only the variation of $\mathcal{V}^{++}$ corresponding to the actual gauge group G. With all such modifications in mind, eq. (16) still holds, and we can use it for computing the variation of the effective action.

Since we now have a large mass parameter in the theory, the effective action can be computed in the most traditional manner, as an expansion in inverse powers of m, with the generic non-Abelian gauge field. Let us represent $\widehat{\Box} = \tilde{\mathcal{O}} - (\mathcal{W}_0 + \mathcal{W})^2 = \mathcal{O} - \mathcal{W}_0{}^2 = \mathcal{O} - m^2$. Then we can represent the operator $\exp(\mathrm{i}s\widehat{\Box})$ in (16) as

$$\mathrm{e}^{\mathrm{i}\,s\,\widehat{\Box}} = \mathrm{e}^{-\mathrm{i}\,s\,m^2}\mathrm{e}^{\mathrm{i}\,s\mathcal{D}^{\hat{a}}\mathcal{D}_{\hat{a}}}\Big\{1 - \frac{\mathrm{i}\,s}{4}(\mathcal{D}^+\mathcal{D}^+\mathcal{W})\,\mathcal{D}^{--} + \frac{(\mathrm{i}\,s)^2}{2}\,(\mathcal{D}^{+\hat{\beta}}\mathcal{W})(\mathcal{D}^{+\hat{\alpha}}\mathcal{W})\,\mathcal{D}^-_{\hat{\alpha}}\mathcal{D}^-_{\hat{\beta}}\Big\} + \dots$$

This will lead to

$$\begin{aligned}\langle J^{++}_\epsilon\rangle &= \mathrm{i}\int_0^\infty \mathrm{d}s\,\frac{(\mu^2 s)^\epsilon}{(4\pi s)^{5/2}}\Big\{\frac{1}{4}(\mathcal{D}^+\mathcal{D}^+\mathcal{W})\,s - (\mathcal{D}^{+\hat{\alpha}}\mathcal{W})(\mathcal{D}^+_{\hat{\alpha}}\mathcal{W})\,(\mathcal{W}_0+\mathcal{W})\,s^2\Big\}\,\mathrm{e}^{-s\,m^2} + \dots \\ &= -\mathrm{sign}(\mathcal{W}_0)\,\frac{\mathrm{i}}{2(4\pi)^2}\Big\{\frac{1}{2}(\mathcal{D}^+\mathcal{D}^+\mathcal{W})\,\mathcal{W}_0 + (\mathcal{D}^{+\hat{\alpha}}\mathcal{W})(\mathcal{D}^+_{\hat{\alpha}}\mathcal{W})\,(\mathcal{W}_0+\mathcal{W})\,\mathcal{W}_0{}^{-1}\Big\} + \dots\end{aligned}$$

By construction, the effective action should depend on $\mathcal{W}_0$ only through the combination $(\mathcal{W}_0 + \mathcal{W})$. However, this structure has been spoiled by our calculational scheme which requires us to Taylor expand contributions with $(\mathcal{W}_0 + \mathcal{W})^2$ at the point $\mathcal{W}_0^2 = m^2$. But the same scheme clearly indicates how one can restore the required structure in the final expression. This is similar to the approach used in [24]. We end up with

$$\langle J_\epsilon^{++} \rangle = -|\mathcal{W}_0| \frac{\mathrm{i}}{64\pi^2} (\mathcal{D}^+\mathcal{D}^+\mathcal{W}) - \mathrm{sign}(\mathcal{W}_0) \frac{1}{32\pi^2} \mathcal{G}^{++} + \dots \ , \tag{43}$$

with $\mathcal{G}^{++}$ defined in (29). As in [24], our final result lacks uniqueness to the extent that we ignore some commutator terms which should be treated as higher order quantum corrections.

The first term on the right of eq. (43) generates the super Yang-Mills term, see [5] for the relevant details, while the second term corresponds to the super Chern-Simons action, in accordance with our previous discussion.

The one-loop calculation performed can also be carried out using either the hybrid superspace approach or projective supergraph techniques [25]. The hypermultiplet action in 4D superspace has the Fayet-Sohnius form [2]

$$S = \int \mathrm{d}^5x \left\{ \int \mathrm{d}^4\theta \, (Q^\dagger \mathrm{e}^{\mathcal{V}} Q + \tilde{Q} \, \mathrm{e}^{-\mathcal{V}} \tilde{Q}^\dagger) + \Big(\int \mathrm{d}^2\theta \, \tilde{Q}(\Phi + m - \partial_5) Q + \mathrm{c.c.} \Big) \right\} \ . \tag{44}$$

Here the chiral superfields Q and $\tilde{Q}$ describe the hypermultiplet, while the adjoint gauge V and chiral Φ superfields correspond to the background 5D vector multiplet. To compute the hypermultiplet effective action, one can use powerful 4D $\mathcal{N} = 1$ functional techniques, see e.g. [26]. The hypermultiplet action in 5D projective superspace [5] is

$$S = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}w}{w} \int \mathrm{d}^5x \, \mathrm{d}^4\theta \, \breve{\Upsilon}(w) \, \mathrm{e}^{\mathcal{V}(w) + \mathcal{V}_0(w)} \, \Upsilon(w) \ . \tag{45}$$

Here the hypermultiplet is described by an arctic superfield $\Upsilon(w)$ and its conjugate, and the 5D vector multiplet is described by a tropical superfield $\mathcal{V}(w)$, see [25, 5] for more details.

Recently, it has been claimed [27] that projective supergraph techniques are more efficient than the harmonic ones. So far, this claim does not seem to have much evidence to support it. Conceptually, the projective supergraphs are essentially equivalent to the harmonic ones [28]. In terms of factual evidence, a great many covariant supergraph calculations for 4D $\mathcal{N} = 2$ super Yang-Mills theories have been carried out within the harmonic superspace approach, see e.g. [14, 15, 16], whereas there has appeared only one non-trivial calculation [29] based on the use of the projective supergraphs. Therefore, it would be very interesting to compute a one-loop low-energy effective action for the theory (45) and try to extract from it a non-Abelian supersymmetric Chern-Simons action realized in terms of the tropical prepotential $\mathcal{V}(w)$ (the Abelian version was given in [5]).

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